Chapter-III

ATMOSPHERIC WAVES:

Wave may be defined as a form of disturbance in a medium.

When a disturbance is given to a part of an elastic medium, then that part gets displaced from its original position. But by the virtue of elasticity, a restoring force is developed in the displaced part, which helps it to return to its original position. This leads to an oscillatory motion, which is known as wave.

Some useful concepts on waves:

WAVE LENGTH:

It is defined as the distance between two consecutive points on the wave, which are in the same phase of oscillation, i.e. distance between two successive troughs or ridges.

WAVE NUMBER:

Wave number of a wave with wave length 'L' is defined as the number of such waves exist around a circle of unit radius. Hence wave number k is defined by, $k = \frac{2\pi}{L}$, where L is the wave length.

Since a wave may travel in any direction, hence we may define wave length / wave number for three directions, viz. along x, y and z directions.

If L_x, L_y and L_z are respectively the wave lengths along x, y and z directions and if k, l and m are wave numbers along x, y and z directions, then

$$k = \frac{2\pi}{L_x},\ l = \frac{2\pi}{L_y} \text{ and } m = \frac{2\pi}{L_z}.$$

FREQUENCY :

It is the number of wave produced in one second.It is denoted by ν.

PHASE VELOCITY:

We know that any disturbance behaves like a carrier. So, wave may be thought of as a carrier. Phase velocity is defined as the rate at which momentum is being carried by the wave. For practical purpose, it may be taken as the speed with which a trough / ridge moves.

It can be shown that, phase velocity in any direction = frequency / wave number in that direction. Thus if the phase velocity vector $\vec{C}$ has components C_x, C_y, C_z along x, y and z direction, then $C_x = \nu / k, C_y = \nu / l$ and $C_z = \nu / m$.

GROUP VELOCITY :

It is the rate at which energy is being carried by the wave. When a single wave travels then the energy and momentum are carried by the wave at the same rate. But when a group of wave travel then momentum propagation rate and energy propagation rates are different. So, in such case group velocity and phase velocity are different. Thus if the phase velocity vector $\vec{C}_G$ has components C_{GX}, C_{GY}, C_{GZ} along x, y and z direction, then $C_{GX} = \frac{\partial \nu}{\partial k}, C_{GY} = \frac{\partial \nu}{\partial l}$ and $C_{CZ} = \frac{\partial \nu}{\partial m}$.

DISPERSION RELATION :

It is a mathematical relation $\nu = f(k,l,m)$ between the frequency (ν) and wave numbers k,l,m.

Generally for any wave, phase velocity and group velocity is obtained from the dispersion relation.

If for any wave phase velocity and group velocity are same, then it is called a non-dispersive wave, otherwise it is a dispersive wave.

ROSSBY WAVE:

First it will be shown how conservation of absolute vorticity (ζ+f) leads to wave like motion.

We consider an object placed on or over the earths surface at latitude ‘φ’ . In the adjoining figure, a meridional circle passing through the object has been shown. Suppose, while motion, the absolute vorticity (ζ+f) of the object remains conserved. Let the object be at stationary state initially.

Then the relative vorticity (ζ) of the object is zero at the initial state. Let f_1 be the value of planetary vorticity at the initial state. Now if the object be displaced meridionally, then its relative vorticity will change to ζ_f (say). If f_2 be the value of planetary vorticity (f) at the final state, then we must have

$$0 + f_1 = \varsigma_f + f_2 \Rightarrow \varsigma_f = -(f_2 - f_1) = -\delta f = -\frac{\partial f}{\partial y}\delta y = -\beta\delta y .$$

Hence, $\varsigma_f > 0$, if δy < 0, i.e., for a southward displacement and

$\varsigma_f < 0$, if δy > 0, i.e., for a northward displacement.

So, if the object is displaced northward, then it turns anti-cyclonically towards its initial latitude. At the initial latitude ζ_f = O, but by inertia it will continue to move southward, cross the initial latitude and acquire cyclonic vorticity. After acquiring cyclonic vorticity, the object turns towards its original latitude. Thus the object executes wave like motion about its initial latitude 'φ' . This wave is known as Rossby wave.

Thus the dynamical constraint for Rossby wave is the conservation of absolute vorticity.

So, to obtain the dispersion relation for the Rossby wave, the governing equation is conservation of absolute vorticity , i.e.

$$\frac{d(\varsigma + f)}{dt} = 0 \Rightarrow \frac{\partial \varsigma}{\partial t} + \vec{V}.\vec{\nabla}\varsigma + v\beta = 0 \quad \text{.....(1)}$$

The above equation is linearised using perturbation method. Here we made the following assumptions :

- Atmosphere is auto-barotropic
- Basic flow is zonal
- Basic zonal flow is meridionally uniform

With these assumptions, the above governing equation may be linearised to

$$\frac{\partial \varsigma'}{\partial t} + U\frac{\partial \varsigma'}{\partial x} + v'\beta = 0 \;\ldots.(2)$$

ς' is perturbation relative vorticity and v' v' is perturbation meridional wind.

For equation (2) we seek for wave like solution, like, $(\;)' \propto e^{i(kx-ct)}$, where, k is the zonal wave number and c is the zonal phase velocity.

After simplification we obtain following dispersion relation , $\nu = U\,k - \frac{\beta}{k}$.

Hence phase velocity C = $\frac{\nu}{k} = U - \frac{\beta}{k^2}$ and group velocity C_G = $\frac{\partial \nu}{\partial k} = U + \frac{\beta}{k^2}$.

Clearly $C \neq C_G$. So, Rossby wave is a dispersive wave.

Since $C - U = -\frac{\beta}{k^2}$, hence Rossby wave retrogates with respect westerly mean flow. Again $C_G - U = \frac{\beta}{k^2} > 0$. Hence Rossby wave carries energy in the downwind direction with respect to westerly mean flow. Physically the above results may be interpreted as follows: For momentum source is the westerly mean flow and for energy the source is the disturbance i.e., the wave.

HAURWITZ WAVE :

This wave is a generalization of the Rossby wave. Similar to Rossby wave, this wave also results from the conservation of absolute vorticity. To obtain the dispersion relation for this wave we take the same assumptions as in Rossby wave except that, here we assume that the basic zonal flow ‘U’ is not uniform in the meridional direction, rather it is a function of ‘y’ (latitude) and amplitude of this wave is zero at y = ± d, i.e., U(± d) = 0.

Starting with the conservation of absolute vorticity, and following the approach, similar to that, made in Rossby wave, we arrive at the following dispersion relationship.

$$\nu = U\,k - \frac{\left(\beta - \dfrac{d^2U}{dz^2}\right)k}{\left(k^2 + \dfrac{\pi^2}{4d^2}\right)}$$

Clearly if the zonal basic flow ‘U’ is uniform in the meridional direction, then $\frac{d^2U}{dz^2} = 0$ and d $\rightarrow \infty$. In that case $\nu = U\,k - \frac{\beta}{k}$ .This is nothing but the dispersion relationship for Rossby wave. So, the Haurwitz wave is a generalisation of Rossby wave.

GRAVITY WAVE

We have seen that to generate any wave always a restoring force is required. Gravity waves are those waves, for which he restoring force is buoyancy.

Classification of Gravity waves:

Gravity waves

External Gravity Wave	Internal Gravity Wave
Can travel along the interface between between two fluids of different densities.	Can travel along the interface between two fluids of different densities.
Cannot travel across the fluid.	Can also travel across the fluid.

Vertical scale is negligible Compared to horizontal scale of motion	Vertical scale is comparable to the horizontal scale of motion
Eg. Sea waves, Tsunami	Eg. Mountain wave, etc.

EXTERNAL GRAVITY WAVE (EGW)

To study the external gravity wave, we consider two different fluids of densities ρ_1 and ρ_2 (ρ_1 > ρ_2) placed one over the other. In the undisturbed condition their interface is a plane surface whose vertical section is a horizontal line as shown in figure 3(a). Now if any perturbation is given to the interface, then it would no longer be a plane surface, rather a wavy surface. Its vertical section would be a wave as shown in fig. 3(b). To study this wave, we consider wave motion in the x-z (Zonal-vertical) plane, as shown in fig. 3(c).

The governing equations are:

- u-momentum equation,
- continuity equation.

These equations are linearized using perturbation method. Then wave like solution is sought for the perturbation height of the interface. Then after simplification we obtain the following dispersion relation.

$\nu = Uk \pm k\sqrt{gH\frac{\Delta\rho}{\rho_1}}$, where, H is the mean depth of the free surface, $\Delta\rho = \rho_1 - \rho_2$.

Now if we take air over ocean water, then definitely $\rho_1 >> \rho_2$ and $\Delta\rho = \rho_1 - \rho_2 \approx \rho_1$, and in that case $\nu = Uk \pm k\sqrt{gH}$

Hence, phase speed $C = \frac{\nu}{k} = U \pm \sqrt{gH}$

And group velocity $C_G = \frac{\partial \nu}{\partial k} = U \pm \sqrt{gH}$.

Hence C = C_G

So, EGW is a non-dispersive wave.

Here $\sqrt{gH}$ is known as shallow water gravity wave speed and U is known as Doppler shift.

Internal gravity wave (IGW) :

To study IGW we consider, for simplicity, a flow which is,

- 2 – D (x-z)
- Adiabatic
- Frictionless
- Non – rotational
- Boussinesq.

The governing equations are:

- U-momentum equation
- W-momentum equation
- Continuity equation
- Energy equation under adiabatic condition.

The above equations are linearised using perturbation method.

The linearised form of the above equations are :

$$\frac{\partial u'}{\partial t} + \overline{U}\frac{\partial u'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial w'}{\partial t} + \overline{U}\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\,\frac{\theta'}{\theta_0}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial \theta'}{\partial t} + \overline{U}\frac{\partial \theta'}{\partial x} = 0$$

Wave solutions, for the perturbations in the above equations are sought. Wave solutions are like $\exp\left[i(kx + mz - \nu t)\right]$

Then after same simplifications we obtain the following dispersion relationship

$$\nu = \overline{U}k \pm \frac{Nk}{\sqrt{k^2 + m^2}} .$$

Phase velocity :

X-Component of phase velocity $C_x = \frac{\nu}{k} = \overline{U} \pm \frac{N}{\sqrt{k^2 + m^2}}$,

Z-Component of phase velocity $C_z = \frac{\nu}{m} = \frac{\overline{U}k}{m} \pm \frac{Nk}{m\sqrt{k^2 + m^2}}$.

Group velocity:

X-Component of group velocity $C_{Gx} = \frac{\partial \nu}{\partial k} = \overline{U} \pm \frac{Nm^2}{\sqrt{k^2 + m^2}}$,

Z-Component of group velocity $C_{Gz} = \frac{\partial \nu}{\partial m} = -\left(\pm \frac{Nkm}{\sqrt{k^2 + m^2}} \right)$.

Now we consider a special case for U = 0

Then $C_z = \pm \frac{Nk}{m\sqrt{k^2 + m^2}}$ and $C_{Gz} = -\left(\pm \frac{Nkm}{\sqrt{k^2 + m^2}} \right)$

Thus it follows that for a given combination of signs of $k, l, ; C_Z$ and C_{GZ} are opposite to each other. Thus vertical phase propagation (momentum propagation) and group propagation (energy propagation) by IGW are opposite to each other.

Also from the above expressions of C's and C_G 's it follows that the vector $\vec{C} = \hat{i}\, C_X + \hat{j} C_Y$ is perpendicular to the phase lines $kx + mz - \nu\, t =$ constant, where as the vector $\vec{C}_G = \hat{i}\, C_{GX} + \hat{j} C_{GY}$ is parallel to the phase lines $kx + mz - \nu\, t =$ constant.

Hence for the IGW, phase velocity and group velocity are perpendicular to each other.

Importance of IGW:

IGW, although, is generated at lower troposphere, they can transport energy, momentum etc upto a great height. From the expressions for phase velocity and group velocity, it is seen that a vertically propagating IGW extracts Westerly Momentum from the mean flow at upper level or imparts easterly momentum to the mean flow at upper level.

IGW is believed to be one of the causes responsible for QBO. CAT is believed to be also due to continuous extraction of momentum from upper level mean flow by